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TWO-SCALE HOMOGENIZATION AND MEMORY EFFECTS OF A FIRST ORDER DIFFERENTIAL EQUATION

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Abstract. We apply the two-scale convergence method introduced by G. Nguetseng and G. Allaire to study the homogenization of a first order linear differential equation. We show that it generates memory effects and the memory kernel is described by a Volterra integral equation. The explicit form of the memory kernel is given in terms of a Radon measure.

1. Introduction

In this paper we consider the homogenization of a first order differential equation

$$\begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} + a^\varepsilon(x,t)u^\varepsilon(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T) \\ u^\varepsilon(x,0) = 0\,, \quad x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N, $f(x,t) \in L^\infty((0,T); L^p(\Omega))$, $1 < p \leq \infty$, $a(x,y,t) \in C(\Omega \times Y \times (0,T))$, Y-periodic in y, and $a^\varepsilon(x,t) = a(x, \frac{x}{\varepsilon}, t)$ satisfying

$$\begin{aligned} &\alpha \leq a^\varepsilon(x,t) \leq \beta\,, && \text{a.e. in } \Omega \times (0,T), \\ &a^\varepsilon \overset{w}{\rightharpoonup} a^0 && \text{weakly-star in } L^\infty(\Omega \times (0,T)), \end{aligned} \tag{1.2}$$

and is equicontinuous in t, i.e., there is a function φ such that $\varphi(\sigma) \to 0$ as $\sigma \to 0$ and

$$|a^\varepsilon(x,t) - a^\varepsilon(x,s)| \leq \varphi(|t-s|) \tag{1.3}$$

We will apply the two-scale convergence method, introduced by G. Nguetseng in [16, 17] and G. Allaire in [1, 2] to study the asymptotic behaviour of (1.1) as ε tends to zero.

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Roughly speaking, the homogenization is a rigorous version of what is known in physics or mechanics as averaging. In other words, homogenization extracts homogeneous effective parameters from disorder or heterogeneous media. Therefore, we will deal with sequences $\{a^\varepsilon\}_\varepsilon$ which describe microscopic quantities and macroscopic quantities are limits of sequences for a suitable weak topology. The homogenization theory studies the behavior of the solution sequence $\{u^\varepsilon\}_\varepsilon$ as $\epsilon \to 0$ and asks whether average behavior can be discerned from differential equations that are subject to high-frequency fluctuations when those fluctuations result from a dependence on two widely separated spatial scales.

The basic idea of the two-scale convergence method is to consider the behaviour of the homogenization process not only from the macroscopic point of view, but also from the microscopic one by introducing an additional microscopic variable y. To this end, test functions of the type $\psi(x, y)$, where ψ is a smooth function, $[0, 1]^N$-periodic in y, are considered. Passing to the two-scale limit, the so called two-scale homogenized problem is obtained; it is usually of the same type as the original problem, but it involved two variables x and y. The classical homogenized problem is then obtained by averaging with respect to y, but in this process, the nice form of the two-scale homogenized problem can disappear. For an introduction of homogenization, especially on the two scale homogenization, to the progress of some current researches and applications we referred to [3] for material science and [15] for the porous media.

When the coefficients satisfy (1.2)-(1.3) and are positive and bounded away from zero such that

$$a^\varepsilon \xrightarrow{w} a^0 \qquad \text{weakly star in } L^\infty(\Omega \times (0, T)),$$

the classical homogenization problem had been discussed by Tartar in [21]. He also proposes an analysis in the case of small amplitude oscillations. For the complete analysis and review we referred to [5] especially for the case when a^ε is independent of t, $a^\varepsilon(x, t) = a^\varepsilon(x)$. Our main results read as follows

Theorem 1.1. *Under hypothesis* (1.2)-(1.3) *and* $1 < p \leq \infty$*, given* $f \in L^\infty((0, T); L^p(\Omega))$ *there exists* $u^0(x, t) \in W^1((0, T); L^p(\Omega))$ *such that the homogenized equation of the equation* (1.1) *is*

$$\left\{ \begin{array}{ll} \dfrac{\partial}{\partial t} u^0(x, t) & + a^0(x, t) u^0(x, t) - \displaystyle\int_0^t K(x, s, t) u^0(x, s)\, ds = f(x, t) \\ u^0(x, 0) & = 0, \quad x \in \Omega\,, \end{array} \right. \tag{1.4}$$

where the memory kernel $K(x, s, t)$ *is given by*

$$K(x, s, t) = \frac{\partial}{\partial s} D(x, s, t) - a^0(x, s) D(x, s, t)\,. \tag{1.5}$$

Here the kernel D defined on $\Omega \times (0,T) \times (0,T)$ is the solution of the Volterra equation

$$(1.6)\qquad \begin{cases} D(x,s,t) = C(x,s,t) + \displaystyle\int_s^t C(x,s,\sigma)D(x,\sigma,t)\,d\sigma. \\ D(x,t,t) = 0\,, \end{cases}$$

with the fluctuation C defined by

$$(1.7)\qquad C(x,s,t) = \int_Y [a(x,y,t) - a^0(x,t)]e^{-\int_s^t a(x,y,\tau)d\tau}dy\,,$$

where $a^0(x,t)$ is the weak star limit of $\{a^\varepsilon\}$ in $L^\infty(\Omega \times (0,T))$, $a(x,y,t)$ is two-scale limit of $\{a^\varepsilon\}$ and $Y = [0,1]^N$.

Furthermore, the memory kernel K given by (1.5) can be represented explicitly with the help of the Radon measure.

Theorem 1.2. *There exists a nonnegative measure μ such that the memory kernel K of* (1.5) *is given by*

$$(1.8)\qquad \begin{aligned} &K(x,s,t) \\ &= \int_Y [a(x,y,t) - a^0(x,t)][a(x,y,s) - a^0(x,s)]e^{-\int_s^t a(x,y,\tau)d\tau}d\mu(y) \end{aligned}$$

for some positive measure $d\mu(y)$.

2. Proof of the Main Theorems

This section is devoted to the proofs of Theorem 1.1 and Theorem 1.2. Before getting into the heart of the matter, we introduce or recall some notations that we shall use in the derivation of the convergence properties. We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbf{R}^N$ that are periodic of period Y. Then $L^p_\#(Y)$, $1 < p \le \infty$, is the completion for the norm of $L^p(Y)$. As stated in the introduction, the two-scale convergence was introduced by G. Nguetseng [16] and G. Allaire [2] in order to obtain an efficient method to study the homogenization problem. A sequence of functions $\{u^\varepsilon\}_\varepsilon \subset L^p(\Omega)$, $1 < p \le \infty$, is said to two-scale converge to a limit $u(x,y) \in L^p(\Omega \times Y)$ if for any function $\psi(x,y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)dx = \int_\Omega \int_Y u(x,y)\psi(x,y)dydx\,.$$

The following theorem states the compactness of two-scale convergence which is the direct extension of the L^2 case. It is the main tool for proving Theorem 1.1.

Theorem 2.1. *For each bounded sequence* $\{u^\varepsilon\}_\varepsilon$ *in* $L^p(\Omega)$, $1 < p \leq \infty$, *there exists a subsequence still denoted by* $\{u^\varepsilon\}_\varepsilon$ *which two-scale converges to* $u(x,y) \in L^p(\Omega \times Y)$.

To establish Theorem 2.1, we need the following lemma. The proof is similar to the L^2 case as given by Allaire in [2] with modification (see also [3], [15]). Therefore, the proof is omitted.

Lemma 2.2. *Let* $\psi(x,y) \in L^q(\Omega; C_\sharp(Y))$, $1 \leq q < \infty$, *then for* $\varepsilon > 0$ $\psi(x, x/\varepsilon)$ *is measurable in* Ω, *we have*

$$\left\|\psi\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^q(\Omega)} \leq \|\psi(x,y)\|_{L^q(\Omega; C_\sharp(Y))} \equiv \left[\int_\Omega \sup_{y \in Y} |\psi(x,y)|^q dx\right]^{\frac{1}{q}}$$

Moreover, it follows that if $\psi(x,y) \in L^q(\Omega; C_\sharp(Y))$ *then*

$$\lim_{\varepsilon \to 0} \int_\Omega \psi^q\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y \psi^q(x,y)\, dy dx$$

and $\psi\left(x, \frac{x}{\varepsilon}\right)$ *two-scale converges to* $\psi(x,y)$.

Proof of Theorem 1.1. We represent the solutions of the nonhomogeneous differential equation (1.1) with the help of the Green's function as

$$u^\varepsilon(x,t) = \int_0^t A^\varepsilon(x,s,t) f(x,s)\, ds\,, \tag{2.1}$$

where

$$A^\varepsilon(x,s,t) \equiv \exp\left(-\int_s^t a(x, x/\varepsilon, \tau) d\tau\right). \tag{2.2}$$

Since $A^\varepsilon \in L^\infty(\Omega \times (0,T))$ and so $A^\varepsilon(x,s,t)\psi(x,x/\varepsilon)$ is bounded in $L^1(\Omega; C_\sharp(Y))$ for all test function $\psi(x,y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$ and for all fixed but arbitrary s and t, Lemma 2.2 implies the existence of $A \in L^\infty(\Omega \times Y)$ for $s,t \in (0,T)$; such that, up to a subsequence, A^ε two-scale convergences to A; that is

$$\begin{aligned}
&\int_\Omega \exp(-\int_s^t a^\varepsilon(x,\tau) d\tau)\psi(x,x/\varepsilon) dx \longrightarrow \\
&\int_\Omega \int_Y A(x,y,s,t)\psi(x,y) dy dx \\
&= \int_\Omega \int_Y \exp(-\int_s^t a(x,y,\tau) d\tau)\psi(x,y) dy dx
\end{aligned} \tag{2.3}$$

Next, we define the functions $B^\varepsilon \in L^\infty(\Omega \times (0,T) \times (0,T))$ by

$$B^\varepsilon(x,s,t) = a^\varepsilon(x,t)A^\varepsilon(x,s,t) = a(x,x/\varepsilon,t)\exp(-\int_s^t a(x,x/\varepsilon,\tau)d\tau) \tag{2.4}$$

in $\Omega \times (0,T) \times (0,T)$. Thus, for $\psi(x,y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$, it is easy to see that $B^\varepsilon\psi$ is also in $L^1(\Omega; C_\sharp(Y))$, again using Lemma 2.2, up to a subsequence, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega B^\varepsilon(x,s,t)\psi\Big(x,\frac{x}{\varepsilon}\Big)dx \\
&\quad = \int_\Omega\int_Y a(x,y,t)\exp\Big(-\int_s^t a(x,y,\tau)d\tau\Big)\psi(x,y)dydx \\
&\quad \equiv \int_\Omega\int_Y B(x,y,s,t)\psi(x,y)dxdy
\end{aligned} \tag{2.5}$$

This means that there exists $B = a(x,y,t)A(x,y,s,t) \in L^\infty(\Omega\times Y)$ for $s,t \in (0,T)$; such that, up to a subsequence, B^ε two-scale converges to B. For $(x,y,s,t) \in \Omega \times Y \times (0,T) \times (0,T)$, we define the function $\widetilde{C}$ by

$$\widetilde{C}(x,y,s,t) \equiv B(x,y,s,t) - a^0(x,t)A(x,y,s,t) \tag{2.6}$$

where $a^0(x,t)$ is the weak star limit of the subsequence of $\{a^\varepsilon(x,t)\}_\varepsilon$ in $L^\infty(\Omega)$ for $t \in (0,T)$. It is clear that u^ε defined by (2.1) satisfies the differential integral equation

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) + \int_0^t a^\varepsilon(x,t)A^\varepsilon(x,s,t)f(x,s)\,ds = f(x,t)\,. \tag{2.7}$$

On the other hand, since $f(x,t) \in L^\infty((0,T); L^p(\Omega))$, we find from (2.1) – (2.3) that there exists $\bar{u}(x,y,t)$;

$$\bar{u}(x,y,t) = \int_0^t A(x,y,s,t)f(x,s)\,ds\,, \tag{2.8}$$

such that, up to a subsequence, u^ε two-scale converges to $\bar{u}(x,y,t) \in L^p(\Omega\times Y)$ for $t \in (0,T)$.

Next, we will study the two-scale convergence of the product of the sequence $\{a^\varepsilon(x,t)u^\varepsilon(x,t)\}_\varepsilon$. Similar argument as B^ε, there exists a function $w^0(x,y,t) \in L^p(\Omega\times Y)$ for $t \in (0,T)$; such that, up to a subsequence, $a^\varepsilon(x,t)u^\varepsilon(x,t)$ two-scale converges to $w^0(x,y,t)$. Clearly, w^0 is given by

$$w^0(x,y,t) \equiv \int_0^t B(x,y,s,t)f(x,s)ds\,. \tag{2.9}$$

Letting ε tend to zero, we find from (2.7) that $\bar{u}$ satisfies the equation

$$\frac{\partial}{\partial t}\bar{u}(x,y,t) + \int_0^t B(x,y,s,t) f(x,s)\,ds = f(x,t) \tag{2.10}$$

in the sense of two-scale convergence. Then substituting (2.6) into (2.10), we derive

$$\frac{\partial}{\partial t}\bar{u}(x,y,t) + \int_0^t \big[\widetilde{C}(x,y,s,t) + a^0(x,t)A(x,y,s,t)\big] f(x,s)\,ds = f(x,t),$$

i.e.,

$$\frac{\partial}{\partial t}\bar{u}(x,y,t) + a^0(x,t)\bar{u}(x,y,t) + \int_0^t \widetilde{C}(x,y,s,t) f(x,s)\,ds = f(x,t)\,. \tag{2.11}$$

Integrating over Y we obtain

$$\frac{\partial}{\partial t}u^0(x,t) + a^0(x,t)u^0(x,t) + \int_0^t C(x,s,t) f(x,s)\,ds = f(x,t)\,.$$

Here $u^0(x,t)$ is the weak limit in $L^p(\Omega)$ for $t \in (0,T)$ of the sequence $\{u^\varepsilon\}$ and

$$C(x,s,t) = \int_Y \widetilde{C}(x,y,s,t)dy. \tag{2.12}$$

To describe the memory or nonlocal kernel, we let

$$g(x,t) \equiv f(x,t) - \int_0^t C(x,s,t) f(x,s)\,ds\,. \tag{2.13}$$

The solution $f(x,t)$ of this Volterra integral equation is given by

$$f(x,t) = g(x,t) + \int_0^t D(x,s,t) g(x,s)\,ds \tag{2.14}$$

where the kernel D solves the resolvent equation

$$D(x,s,t) = C(x,s,t) + \int_s^t C(x,s,\sigma) D(x,\sigma,t)\,d\sigma\,. \tag{2.15}$$

Due to the initial conditions in (2.1), it is easy to see that

$$D(x,t,t) = 0\,, \qquad \partial_t D(x,t,t) = 0\,.$$

It is obvious from (2.15) that the kernel D is bounded and equicontinuous in t and has a bounded derivative in s. In fact

$$\frac{\partial}{\partial s}D(x,s,t) = \frac{\partial}{\partial s}C(x,s,t) + \int_s^t \frac{\partial}{\partial s}C(x,s,\sigma) D(x,\sigma,t)\,d\sigma$$

i.e., D has the same regularity as C. Notice that (2.11) has the same form as (2.14) by letting

$$g(x,t) = \frac{\partial}{\partial t}u^0(x,t) + a^0(x,t)u^0(x,t)\,. \tag{2.16}$$

This implies that the weak limit u^0 satisfies the differential integral equation

$$\begin{aligned}&\frac{\partial}{\partial t}\ u^0(x,t) + a^0(x,t)u^0(x,t)\\&\quad + \int_0^t D(x,s,t)\left[\frac{\partial}{\partial s}u^0(x,s) + a^0(x,s)u^0(x,s)\right] ds = f(x,t)\,,\end{aligned} \tag{2.17}$$

which after being integrated by part and using the initial conditions will become

$$\begin{aligned}&\frac{\partial}{\partial t}\ u^0(x,t) + a^0(x,t)u^0(x,t)\\&\quad - \int_0^t \left[\frac{\partial}{\partial s}D(x,s,t) - a^0(x,s)D(x,s,t)\right]u^0(x,s)\, ds = f(x,t)\end{aligned} \tag{2.18}$$

or

$$\frac{\partial}{\partial t}u^0(x,t) + a^0(x,t)u^0(x,t) - \int_0^t K(x,s,t)u^0(x,s)\, ds = f(x,t) \tag{2.19}$$

where the kernel K is given by

$$K(x,s,t) = \frac{\partial}{\partial s}D(x,s,t) - a^0(x,s)D(x,s,t) \tag{2.20}$$

with $(x,s,t) \in \Omega \times (0,T) \times (0,T)$. This completes the proof of Theorem 1.1.

Remark 2.3. Let us remark that the classical method of asymptotic expansions can also be used to prove this theorem. However, this is at the price of more assumptions on the smoothness of the physical data. Under the same hypothesis, Tartar [21] had proved that there exists a subsequence of $\{a^\varepsilon\}_\varepsilon$ such that $\{u^\varepsilon\}_\varepsilon$ converges in $W^{1,\infty}(0,T; w^*\text{-}L^\infty(\Omega))$ to u^0 satisfying the same homogenized equation. This theorem is, therefore, a generalization of Tartar's.

To obtain the exact form of the kernel, $K(x,s,t)$ we have to figure out the kernel $D(x,s,t)$ in (2.15) first. Indeed, we need the following important lemma.

Lemma 2.4. *There exists a Radon measure μ on Y, such that the solution $D(x,s,t)$ of the resolvent equation* (2.15) *is explicitly given by*

$$D(x,s,t) = \int_Y F(x,y,s,t)d\mu(y) \tag{2.21}$$

where

$$F(x,y,s,t) = \big[a(x,y,t) - a^0(x,t)\big] \exp\left(-\int_s^t a(x,y,\tau)d\tau\right) \tag{2.22}$$

Proof. For any $\phi \in C(\Omega \times Y \times [0,T] \times [0,T]; \mathbf{R})$, we denote by H the set that is

$$H \equiv \{\phi_{\bar{x},\bar{s},\bar{t}} : Y \to \mathbf{R} | \bar{s}, \bar{t} \in [0,T]; \bar{x} \in \Omega\} \equiv \{\phi_{\bar{x},\bar{s},\bar{t}}\} .$$

Let M be the vector space generated by H, then clearly M is a subspace of the space $C(Y)$. Now we define a linear operator $T : M \to \mathbf{R}$ by
(2.23)

$$\langle T, \phi_{\bar{x},\bar{s},\bar{t}}(y)\rangle = \int_Y \phi_{\bar{x},\bar{s},\bar{t}}(y)dy + \int_0^T \chi_{[\bar{s},\bar{t}]}(\bar{\sigma})\Big[\int_Y \phi_{\bar{x},\bar{\sigma},\bar{t}}(y)dy\Big] D(\bar{x},\bar{s},\bar{\sigma})d\bar{\sigma}$$

then

$$|\langle T, \phi_{\bar{x},\bar{s},\bar{t}}(y)\rangle| \le \|\phi\|_{C(Y)} + \|\phi\|_{C(Y)} \cdot C_1 \le C\|\phi\|_{C(Y)}$$

where C is a constant. This shows that T is a bounded functional on M. Then by Hahn-Banach Theorem, there exists a bounded functional Λ on $C(Y)$, such that $\Lambda|_M = T$; therefore using the Riesz representation theorem, we deduce that there exists a Radon measure μ on Y such that

$$\langle \Lambda, \psi\rangle = \int_Y \psi(y)d\mu(y), \qquad \forall \psi \in C(Y). \tag{2.24}$$

Choosing $\psi(y) = \phi_{x,s,t}(y)$, then by (2.23) – (2.24), we obtain

$$\begin{aligned}
&\int_Y \phi_{x,s,t}(y)d\mu(y) = \langle \Lambda, \phi_{x,s,t}(y)\rangle \\
&= \int_Y \phi_{x,s,t}(y)dy + \int_0^T \chi_{[s,t]}(\sigma)\Big[\int_Y \phi_{x,\sigma,t}(y)dy\Big] D(x,s,\sigma)d\sigma.
\end{aligned}$$

In particular, let

$$\phi_{x,s,t}(y) = [a(x,y,t) - a^0(x,t)] \exp\left(-\int_s^t a(x,y,\tau)d\tau\right) \tag{2.25}$$

and use the equations (1.7) and (2.15), we derive the relation

$$\begin{aligned}
&\int_Y \phi_{x,s,t}(y)d\mu(y) \int_Y \phi_{x,s,t}(y)dy + \int_0^T \chi_{[s,t]}(\sigma)\Big[\int_Y \phi_{x,\sigma,t}(y)dy\Big] D(x,s,\sigma)d\sigma \\
&= D(x,s,t)
\end{aligned}$$

equivalently,

$$D(x,s,t) = \int_Y \big[a(x,y,t) - a^0(x,t)\big] e^{-\int_s^t a(x,y,\tau)d\tau} d\mu(y)$$

This completes the proof of the lemma.

We now return to the proof of the Theorem 1.2. We denote the fluctuation function $C(x,s,t)$ by

$$C(x,s,t) = \int_Y \big[a(x,y,t) - a^0(x,t)\big] e^{-\int_s^t a(x,y,\tau)d\tau} dy \tag{2.26}$$

It follows immediately from (2.20), (2.21) and (2.22) that

$$\begin{aligned} &K(x,s,t) \\ &= \textstyle\int_Y \big[a(x,y,t) - a^0(x,t)\big]\big[a(x,y,s) - a^0(x,s)\big] e^{-\int_s^t a(x,y,\tau)d\tau} d\mu(y). \end{aligned}$$

Thus, we have proved Theorem 1.2.

3. Some Examples

In this section we will apply the main theorem to some concrete examples. We first give the explicit characterization for the K when a^ε is of separable variables, $a^\varepsilon(x,t) = \theta^\varepsilon(t)a^\varepsilon(x)$, then the equation (1.1) becomes

$$\begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} + \theta^\varepsilon(t)a^\varepsilon(x)u^\varepsilon(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T) \\ u^\varepsilon(x,0) = 0\,, \quad x \in \Omega\,. \end{cases} \tag{3.1}$$

We suppose that

$$0 < \theta_- \le \theta^\varepsilon(t) \le \theta_+\,, \qquad \theta^\varepsilon \overset{w}{\rightharpoonup} \theta^0 \qquad \text{in } L^\infty(0,T) \text{ weak star} \tag{3.2a}$$

$$0 < a_- \le a^\varepsilon(x) \le a_+\,, \qquad a^\varepsilon \overset{w}{\rightharpoonup} a^0 \qquad \text{in } L^\infty(\Omega) \text{ weak star.} \tag{3.2b}$$

The solutions of the equation (3.1) are given by

$$u^\varepsilon(x,t) = \int_0^t \exp\Big[-a^\varepsilon(x)\Theta^\varepsilon(s,t)\Big] f(x,s)ds \tag{3.3}$$

where Θ^ε is the antiderivative of θ^ε

$$\Theta^\varepsilon(s,t) = \int_s^t \theta^\varepsilon(\tau)d\tau\,. \tag{3.4}$$

Due to the hypothesis $(3.2a)$ and the strong convergence of Θ^ε to

$$\Theta^0(s,t) = \int_s^t \theta^0(\tau)d\tau$$

we deduce that the solution sequence given by (3.3) behaves like

$$v^\varepsilon(x,t) = \int_0^t \exp\Big[- a^\varepsilon(x)\Theta^0(s,t)\Big] f(x,s)ds\,. \tag{3.5}$$

Therefore, to study the homogenization problem of (3.1) is equivalent to consider the following equation

$$\begin{cases} \dfrac{\partial v^\varepsilon(x,t)}{\partial t} + \theta^0(t)a^\varepsilon(x)v^\varepsilon(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T) \\ v^\varepsilon(x,0) = 0\,, \quad x \in \Omega\,. \end{cases} \tag{3.6}$$

We can apply Theorem 1.1 to conclude that:

Theorem 3.1. *After extracting a subsequence, there exists a positive measure $d\mu(y)$ associated with $\{a^\varepsilon\}$, and a kernel $K(x,s,t)$:*

$$K(x,s,t) = \int_Y \theta^0(t)\theta^0(s)\big[a(x,y) - a^0(x)\big]^2 e^{-a(x,y)\int_s^t \theta^0(\tau)d\tau} d\mu(y) \tag{3.7}$$

defined in $\Omega \times (0,T) \times (0,T)$, such that the sequence $\{u^\varepsilon\}_\varepsilon$ converges weakly in $L^\infty((0,T); L^p(\Omega))$, $1 < p \le \infty$, to u^0 which is the solution of

$$\frac{\partial}{\partial t}u^0(x,t) + \theta^0(t)a^0(x)u^0(x,t) - \int_0^t K(x,s,t)u^0(x,s)\,ds = f(x,t)\,. \tag{3.8}$$

We are also interested in the homogenization of the following hyperbolic equation

$$\begin{cases} \partial_t u^\varepsilon(x,t) + b(x,t)\partial_x u^\varepsilon(x,t) + a^\varepsilon(x,t)u^\varepsilon(x,t) = f(x,t) \\ u^\varepsilon(x,0) = 0 \end{cases} \tag{3.9}$$

with $b(x,t) \in C(\Omega \times (0,T))$ and $a^\varepsilon(x,t)$ the same result as given in section 2. The characteristic curve $X(x,t,s)$ is defined, for $s \in (0,T)$ and $x \in \Omega$, by

$$\frac{dX(x,t,s)}{dt} = b(X,t), \qquad t \in (0,T), \qquad X\,|_{t=s} = x. \tag{3.10}$$

Clearly, for $(x,t) \in \Omega \times (0,T)$ the function

$$w^\varepsilon(x,t) = u^\varepsilon(X(x,t,0),t) \tag{3.11}$$

is the solution of the equation

$$\begin{cases} \partial_t w^\varepsilon(x,t) + a^\varepsilon(X(x,t,0),t)w^\varepsilon(x,t) = f(x,t) \\ w^\varepsilon(x,0) = 0 \end{cases} \tag{3.12}$$

where $a^\varepsilon(X(x,t,0),t) := a(X(x,x/\varepsilon,t,0),t)$. By characteristic,

$$w^\varepsilon(x,t) = u^\varepsilon(X(x,t,0),t) = \int_0^t e^{-\int_s^t a^\varepsilon(X(x,\tau,0),\tau)d\tau} f(x,s)ds. \tag{3.13}$$

Passing to the limit as $\varepsilon \to 0$, up to a subsequence, the sequence $\{w^\varepsilon\}$ converges in the sense of two-scale convergence to w given by

$$w(x,t,y) = \int_0^t e^{-\int_s^t a(X(x,y,\tau,0),\tau)d\tau} f(x,s)ds, \tag{3.14}$$

and the weak limit function w^0 of w^ε in $L^p(\Omega)$, given by

$$u^0(X(x,t,0),t) = w^0(x,t) = \int_Y w(x,t,y)dy, \qquad t \in (0,T). \tag{3.15}$$

Now applying the main theorem we deduce that the homogenized equation of (3.12) is given by

$$\partial_t w^0(x,t) + a^0(X(x,t,0),t)w^0(x,t) - \int_0^t K(x,s,t)w^0(x,s) = f(x,t) \tag{3.16}$$

where the kernel K is given by

$$\begin{aligned} K(x,s,t) = \int_Y &\Big[a(X(x,y,t,0),t) - a^0(x,t)\Big] \\ &\Big[a(X(x,y,s,0),s) - a^0(x,s)\Big] \\ &e^{-\int_s^t a(X(x,y,\tau,0),\tau)d\tau} d\mu(y) \end{aligned} \tag{3.17}$$

with $(x,s,t) \in \Omega \times (0,T) \times (0,T)$ for some positive measure $d\mu(y)$. Returning to the (x,t) variables we have proved the following theorem.

Theorem 3.2. *The homogenized equation of* (3.9) *is*

$$\begin{aligned} \partial_t u^0(x,t) \ &+ b(x,t)\partial_x u^0(x,t) + a^0(x,t)u^0(x,t) \\ &- \int_0^t K(x,s,t)w^0(x,s) = f(x,t) \end{aligned} \tag{3.18}$$

where the kernel K is given by (3.17).

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